Approximations of High-Dimensional Markov Perfect Equilibrium in Dynamic Oligopoly

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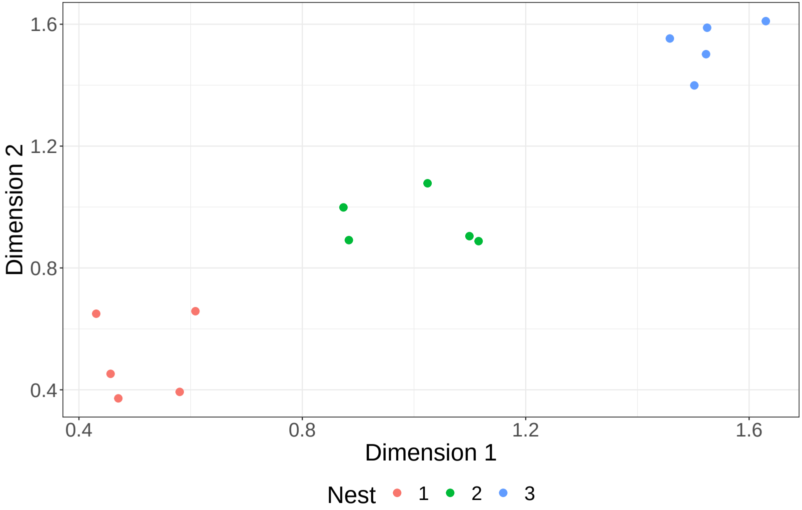
[1]These slides partially rely on material provided by Dean Corbae.

Introduction

- **Motivation**: Models of industry dynamics have a curse of dimensionality
  - Problems become intractable with many firms and high-dimensional heterogeneity
  - Solution concepts mainly allow for low-dimensional heterogeneity
- **Question**: How can we incorporate high-dimensional heterogeneity in dynamic problems?
  - Is an approximation of the state space a viable method? When does it perform well?
  - Hybridize Markov perfect equilibrium and oblivious equilibrium to incorporate many states?
- **Project**: Formalize nested equilibrium concepts and run simulations
  - *Intuition*: Clustering algorithms or *a priori* specification identify similar firms
  - *Current*: Original solution concepts are special cases and static simulations
    - Experiments: separation of nests, nesting parameter, and overlap in nests
  - *Next Steps*: Wrap up the dynamic simulations, application

Simulated Example

Figure: Simulated Data



Roadmap

Roadmap

Related Literature

- Many studies follow the original model in Ericson and Pakes (1995) and algorithm in Pakes and McGuire (1994)
- New solution concepts and approximations address the curse of dimensionality
  - *Asymptotic games*: Weintraub et al. (2008) and Benkard et al. (2015) assume firms respond as if the industry is in the steady state
  - *Continuous time*: Doraszelski and Judd (2012) cast the game in continuous time
  - *Approximations*: Doraszelski (2012) and Barwick and Pathak (2015) approximate the value function with flexible functions
- Recent papers use machine learning to reduce dimensionality and identify clusters
  - Magnolfi et al. (2022) use distances between products in latent space in static demand
  - Raisingh (2022) and Barwick et al. (2021) use an index to summarize a state
  - Atalay et al. (2023) identify nests of products with clustering techniques

Model

Assumptions

- Firms $j \in \{1, ..., J\}$
- Infinite horizon discrete time $t \in \{1, 2, 3, ...\}$ with discount factor $\beta \in (0, 1)$
- State x_{jt} tracks the quality x of firm j in period t resulting in per-period profits $\Pi(x_{jt}, s_{-jt})$
- s_t is the state vector of all firms in period t, i.e., a histogram[2]
    - For example, when $x \in \{0, 1\}$, two firms are at $x = 0$, and three firms are at $x = 1$, then the industry state is $s_t = (s_t(0), s_t(1)) = (2, 3)$
    - This implies the total number of firms is $N_t = \sum_x s_t(x)$
- Curse of dimensionality: with 20 firms and 40 states, there are $\binom{N+K}{K} \approx 2e15$ states

[2] In the extension, firms must also track an index for their respective nests in $\mathbf{s}_{-jt}$.

State Space and Payoffs

- The state space is the set of all industry states

$$\mathcal{S} = \left\{ s \in \mathbb{N}^{\infty} \middle| \sum_{x} s(x) < \infty \right\}$$

- Notably, the state of the competitors of firm j is given by:

$$s_{-jt}(x) = \begin{cases} s_t(x) - 1 & \text{if } x = x_{jt} \\ s_t(x) & \text{otherwise} \end{cases}$$

- Profits are characterized by static Nash equilibrium, e.g., with logit demand:

$$\Pi(x_{jt}, s_{-jt}) = m\sigma(x_{jt}, s_{-jt}, p_t)(p_{jt} - c)$$

Investment, Entry, and Exit

- Firms invest ι to improve future quality which is successful with some probability, e.g.:

$$P(x_{j,t+1}|x_{jt} = x, \iota) = \begin{cases} \frac{(1-\delta)a\iota}{1+a\iota} & \text{if } x_{j,t+1} = x+1 \\ \frac{(1-\delta)+\delta a\iota}{1+a\iota} & \text{if } x_{j,t+1} = x \\ \frac{\delta}{1+a\iota} & \text{if } x_{j,t+1} = x-1 \end{cases}$$

- Quality can stochastically depreciate by one with probability δ
- Incumbent firms privately observe random sell-off values ϕ_{jt}[3]
- Potential entrants have random entry costs κ_{jt} and appear in state x^e
  - The number of entrants is i.i.d. and Poisson with mean $\lambda(s_t)$

[3] This a key difference from Hopenhayn to smooth exit decisions.

Markov Perfect Equilibrium

Timing

- Incumbents have investment strategies ι and exit strategies ρ such that $\mu \equiv (\iota, \rho)$
- Potential entrants have cut-off entry strategies λ
- The value function for some random exit time τ_j is given by:

$$V(x, s|\mu', \mu, \lambda) = \mathbb{E}_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_j} \beta^{k-t}(\Pi(x_{jk}, s_k) - c(\iota_{jk}, x_{jk})) + \beta^{\tau_j - t}\phi_{j,\tau_j} \middle| x_{jt} = x, s_t = s\right]$$

- Equilibrium is composed of strategies μ and λ that satisfy the following conditions:
  1. Incumbent strategies are optimal:

$$\sup_{\mu' \in \mathcal{M}} V(x, s|\mu', \mu, \lambda) = V(x, s|\mu, \lambda), \qquad \forall (x, s) \in \mathcal{X} \times \mathcal{S}$$

  2. The cut-off rule for entry is determined by the expected discounted value of profits in the entry state for any industry state:

$$\lambda(s) = \beta \mathbb{E}_{\mu,\lambda}\left[V(x^e, s_{t+1}|\mu, \lambda)|s_t = s\right], \qquad \forall s \in \mathcal{S}$$

Roadmap

Oblivious Equilibrium

- The curse of dimensionality makes computation intractable
- *Key idea*: Assume the industry is in the steady-state already
    - If there are enough firms, the industry should be close to constant
    - Reasoning along the lines of the law of large numbers
- No longer track *every possible* industry state s_t because it is assumed to be constant

Oblivious Value Function

- Denote the long-run expectation of the industry state $\tilde{s} = \mathbb{E}[s]$
- Firms assume that the current state is the long-run average

$$\lim_{t \to \infty} \tilde{s}_t(x) = \lambda \sum_{k=0}^{\infty} P_\mu^k(x^e, x) \qquad \forall x$$

- This implies that $\tilde{s}_{\mu,\lambda} = \lambda(I - P_\mu)^{-1}$ for entrants
- Oblivious strategies and value function:

$$\tilde{V}(x|\mu', \mu, \lambda) = \mathbb{E}_{\mu'}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}(\Pi(x_{ik}, \tilde{s}_{\mu,\lambda}) - c(\iota_{ik}, x_{ik})) + \beta^{\tau_i - t}\phi_{i,\tau_i} \middle| x_{it} = x\right]$$

Oblivious Equilibrium

1. Incumbent strategies optimize their oblivious value functions:

$$\sup_{\mu' \in \tilde{\mathcal{M}}} \tilde{V}(x|\mu', \mu, \lambda) = \tilde{V}(x|\mu, \lambda), \qquad \forall x \in \mathcal{X}$$

2. Either the expected (oblivious) expected value of entry is zero, the entry rate is zero, or both:

$$\begin{aligned}
&\lambda \left[\beta \tilde{V}(x^e|\mu', \mu, \lambda) - \kappa\right] = 0, \\
&\beta \tilde{V}(x^e|\mu', \mu, \lambda) - \kappa \leq 0, \\
&\lambda \geq 0.
\end{aligned}$$

Dimensionality reduction: $k^J \to k$ states

Comparison to Markov Perfect Equilibrium

Figure: Method Comparison

Table 1: Comparison of MPE and OE strategies (4 firms, no entry and exit)

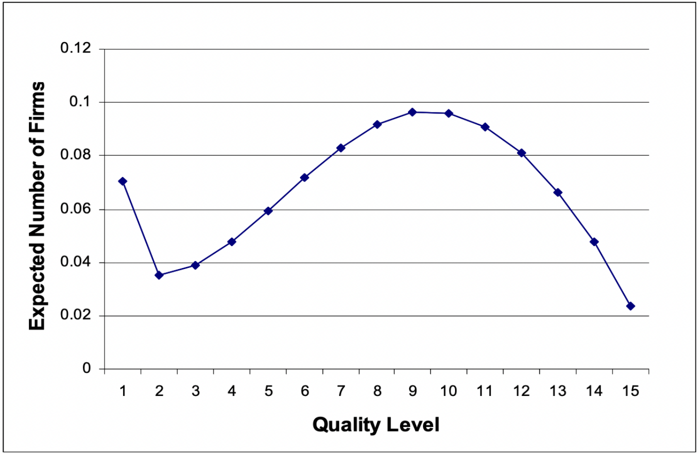
Parameters		Long Run Statistics (% Diff)					Perf Bound (% Diff)		Actual (% Diff)	
θ_1	d	Inv.	Prod Surp	Cons Surp	C1	C2	Max Diff	Weighted Avg	Max Diff	Weighted Avg
0.10	0.10	−0.26	−0.01	−0.02	0.03	0.03	0.14	0.13	0.08	0.07
0.30	0.30	−0.13	0.06	0.08	0.08	0.16	1.67	1.22	0.04	0.01
0.50	0.50	−0.11	0.20	0.28	0.18	0.50	6.64	3.61	0.21	0.06
0.70	0.70	−2.21	0.40	0.15	1.08	2.09	18.85	8.35	1.60	0.67
0.85	0.70	−2.19	0.23	−0.28	1.37	2.10	30.80	9.64	1.80	0.20
0.15	0.27	3.54	0.14	0.2	1.22	0.46	0.36	0.35	0.1	0.1
0.20	0.35	4.18	0.29	0.42	1.93	1.03	0.81	0.77	−0.09	−0.05
0.30	0.55	9.28	0.93	1.31	5.10	2.45	1.96	1.85	0.26	0.25
0.40	0.80	21.02	2.10	2.93	11.58	4.12	3.01	2.92	0.30	0.29
0.50	1.00	18.62	3.30	4.33	15.69	5.94	6.29	5.86	0.32	0.30

Long run statistics and value functions simulated with a relative precision of 1.0% and a confidence level of 99%. Error bound simulated with a relative precision of at most 10% and a confidence level of 99%.

Distribution of States

Figure: Distribution of States

Figure 4: Average industry state for $\theta_1 = 0.5$ and $d = 0.5$.



Partially Oblivious Equilibrium

Value Function

1. Incumbent strategies optimize their partially oblivious value functions:

$$\sup_{\mu' \in \tilde{\mathcal{M}}_p} \tilde{V}_p(\bar{x}, w | \mu', \mu, \lambda) = \tilde{V}_p(\bar{x}, w | \mu, \lambda), \qquad \forall \bar{x}, w$$

2. Either the expected (oblivious) expected value of entry is zero, the entry rate is zero, or both:

$$\sum_w \lambda(w) \left[\beta \mathbb{E}[\tilde{V}_p((x^e, 0), w_{t+1} | \mu', \mu, \lambda) | w_t = w] - \kappa \right] = 0,$$
$$\beta \mathbb{E}[\tilde{V}_p((x^e, 0), w_{t+1} | \mu', \mu, \lambda) | w_t = w] - \kappa \leq 0 \qquad \forall w,$$
$$\lambda(w) \geq 0 \qquad \forall w.$$

Dimensionality reduction: $k^J \to k^{D+1}$ states

Moment-Based Markov Equilibrium
Value Function

1. Incumbent strategies optimize their perceived value functions:

$$\sup_{\mu' \in \tilde{\mathcal{M}}_m} \tilde{V}_m(x, \hat{s}|\mu', \mu, \lambda) = \tilde{V}_m(x, \hat{s}|\mu, \lambda), \qquad \forall x, \hat{s}$$

2. The expected perceived discounted value of entry is equal to the cut-off value for entry:

$$\lambda(\hat{s}) = \beta \mathbb{E}_{\mu,\lambda} \left[\tilde{V}_m(x^e, \hat{s}_{t+1}|\mu, \lambda) | \hat{s}_t = \hat{s} \right], \qquad \forall \hat{s} \in \hat{\mathcal{S}}$$

Dimensionality reduction: $k^J \to k^D$ states

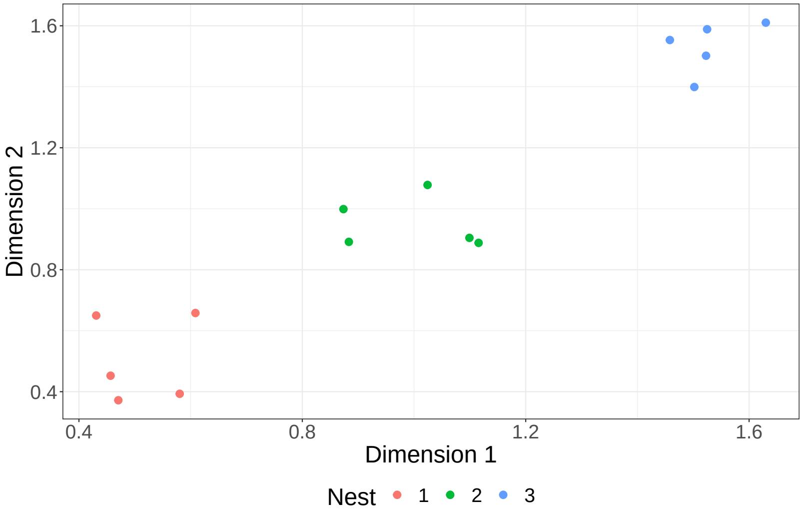
High-Dimensional Heterogeneity

- The state space can expand to include heterogeneity
- However, it is subject to a curse of dimensionality in heterogeneity
- The problem becomes more intractable when there are many continuous dimensions

Roadmap

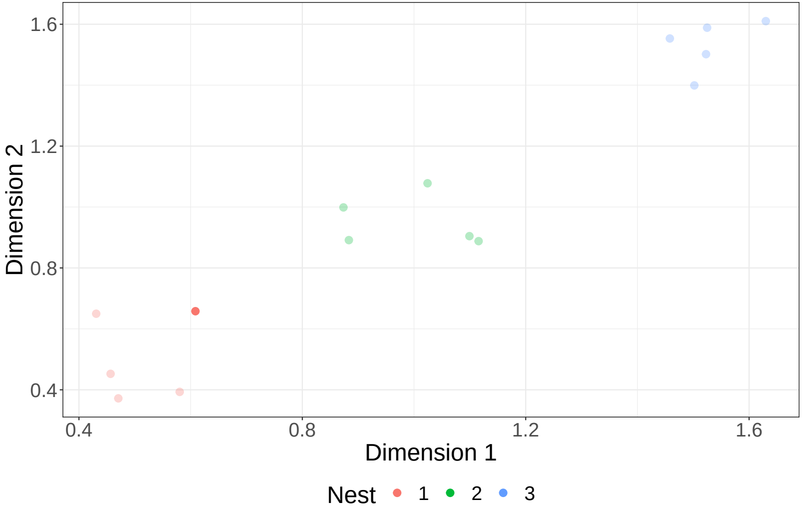
Simulated Example

Figure: Simulated Data



Simulated Example

Figure: Simulated Data



Simulated Example

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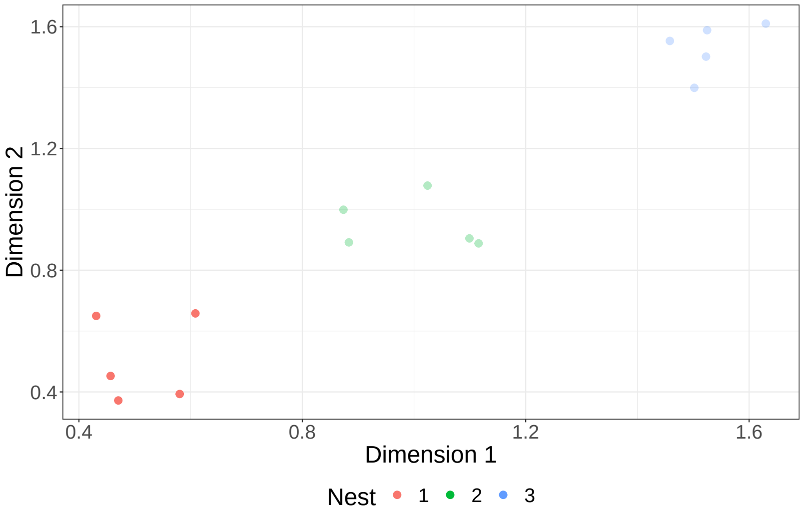

Simulated Example

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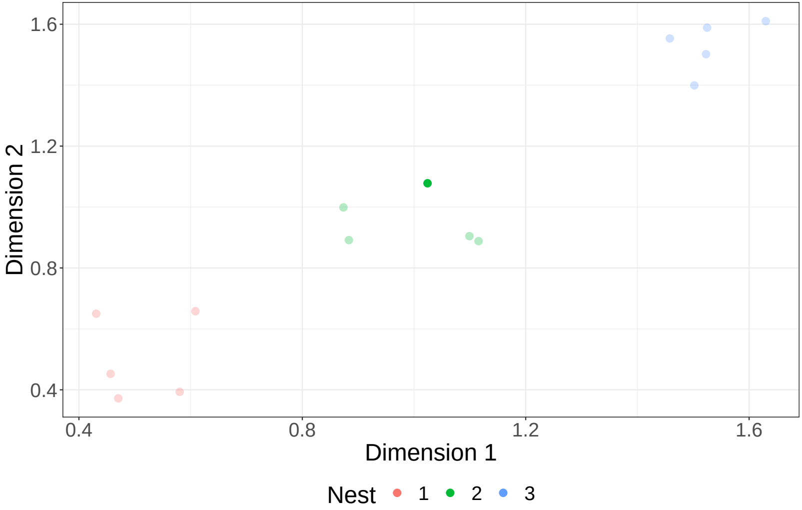

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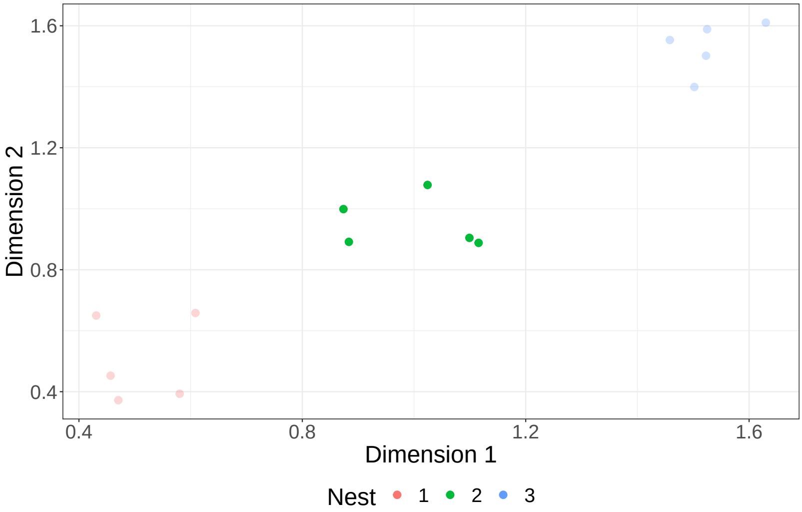

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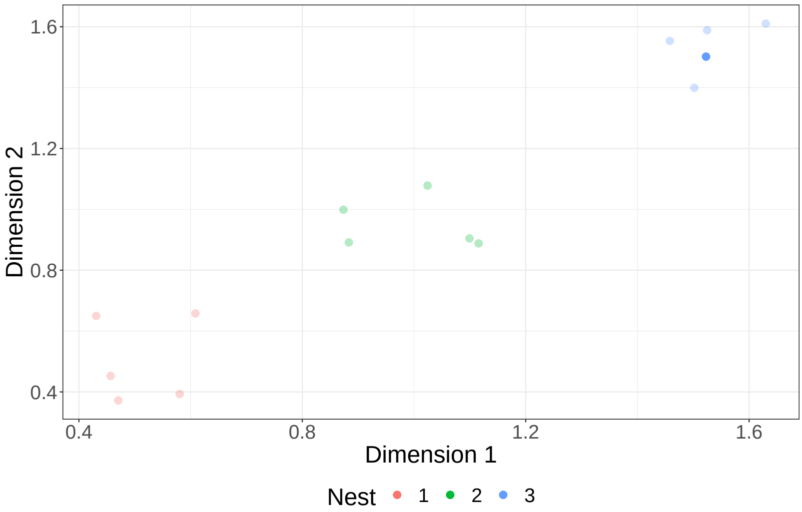

Simulated Example

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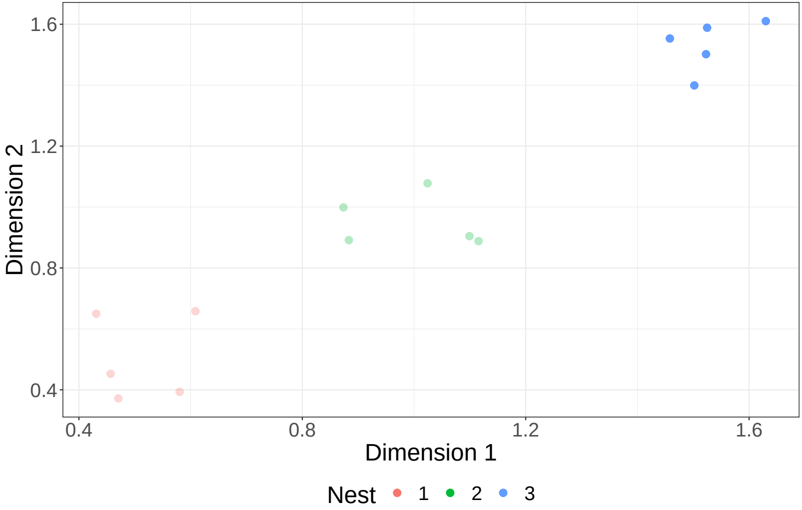


Nested Markov Equilibria

- *Nested Markov equilibria* are the proposed solution concepts
  - Extensions to oblivious, partially oblivious, and moment-based Markov equilibria
- Firms that are close to each other in characteristics space compete more strongly
- Rely on a few key assumptions
  - Nests are recoverable with some function
  - Firms use nests to approximate the state space in their optimization problems
  - Firms can only belong to one nest, i.e., no overlap

Assumptions

Assumption (Recoverable Nests)

The data-generating process for the high-dimensional heterogeneity of firms is based on underlying low-dimensional nests and is unaffected by investment decisions. Further, there exists a function G that recovers the nests from the high-dimensional object $\mathbf{X}$. That is, the nest is determined by $g = G(\mathbf{X})$.

Assumptions

Assumption (Firm State Approximations)

Firms use their respective nests in their optimization problems as a sufficient statistic rather than the full and potentially high-dimensional or continuous state space. Namely, the conditions required for Markov perfect equilibrium are satisfied using a simplified state space that only includes nests $\mathcal{G}$ *rather than a high-dimensional object* $\mathbf{X}$*:*

$$V(x, s(\mathbf{X})|\mu, \lambda) = V(x, s(\mathcal{G})|\mu, \lambda)$$

$$\lambda(s(\mathbf{X})) = \lambda(s(\mathcal{G}))$$

That is, firms optimize over $s(\mathcal{G})$ *rather than* $s(\mathbf{X})$*.*

Assumptions

Assumption (No Overlap)

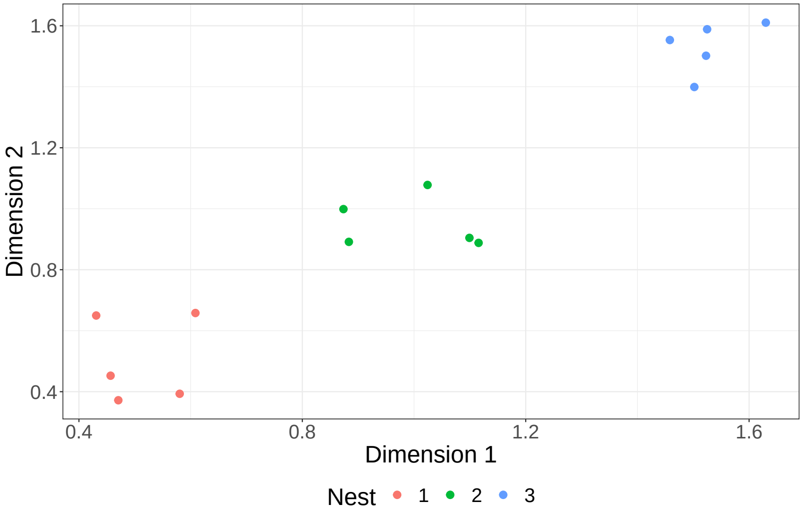
Firms respond symmetrically within their respective nests. This obviates the need to track firm identities and thus restricts the model to an anonymous equilibrium. In other words, nests do not overlap.

Proposed Strategies

- Nested oblivious equilibrium
  1. Nests are independent and internally oblivious
  2. Nests are oblivious to themselves and other nests
  3. Nests are independent and not oblivious to themselves
  4. Nests are oblivious to other nests but not to themselves

Simulated Example

Figure: Simulated Data



Nested Oblivious Equilibrium

Algorithm

Proposition (1)

Cases (i), (ii), and (iv) nest oblivious equilibrium as a special case.

Proof

Proposition (2)

Cases (iii) and (iv) nest Markov perfect equilibrium as a special case.

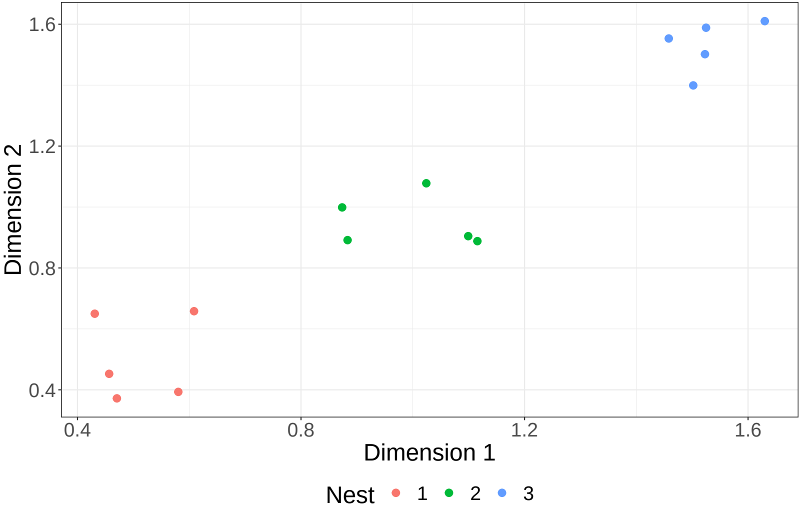
Proof

Proposed Strategies

- Nested oblivious equilibrium
- Nested partially oblivious equilibrium
  1. Nests are independent and partially oblivious to themselves
  2. Nests are partially oblivious to themselves and oblivious to other nests

Simulated Example

Figure: Simulated Data



Nested Partially Oblivious Equilibrium Algorithm

Proposition (3)

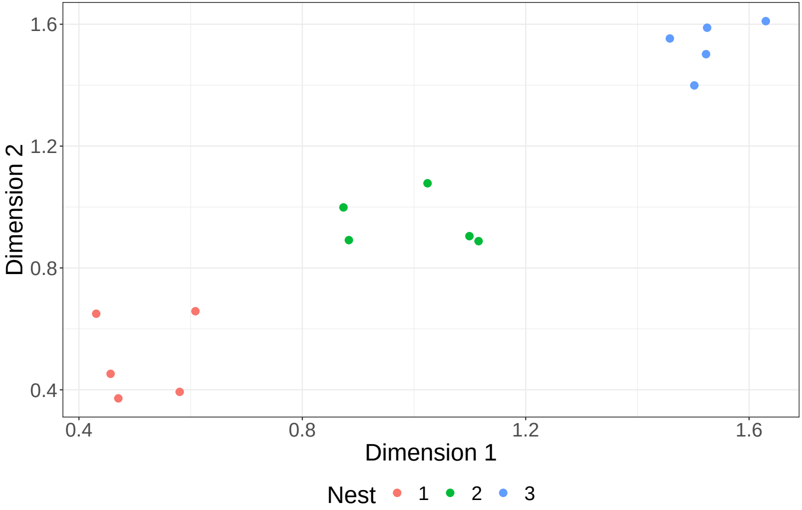
Nested oblivious equilibrium, oblivious equilibrium, partially oblivious equilibrium, and full Markov perfect equilibrium are special cases of nested partially oblivious equilibrium. Proof

Proposed Strategies

- Nested oblivious equilibrium
- Nested partially oblivious equilibrium
- Nested moment-based Markov equilibrium
    1. Nests are independent and firms use moment-based strategies to respond to competitors
    2. Nests use moment-based strategies to respond to competitors within and outside their nests

Simulated Example

Figure: Simulated Data



Nested Moment-Based Markov Equilibrium

Algorithm

Proposition (4)

Nested partially oblivious equilibrium and moment-based Markov equilibrium are special cases of nested moment-based Markov equilibrium. Subsequently, partially oblivious equilibrium, nested oblivious equilibrium, oblivious equilibrium, and Markov perfect equilibrium are special cases as well.

Proof

Roadmap

Static Simulations

- Static simulations show the error introduced by the firm state approximation assumption
- Nested logit is the environment of interest

$$u_{ijt} = \alpha p_{jt} + \beta \mathbf{X}_{jt} + \varepsilon_{ig(j)t} + (1-\rho)\varepsilon_{ijt}$$

- High-dimensional heterogeneity comes in through $\mathbf{X}$
- *Question*: When does the state approximation assumption hold?

Static Simulations Simulated Data

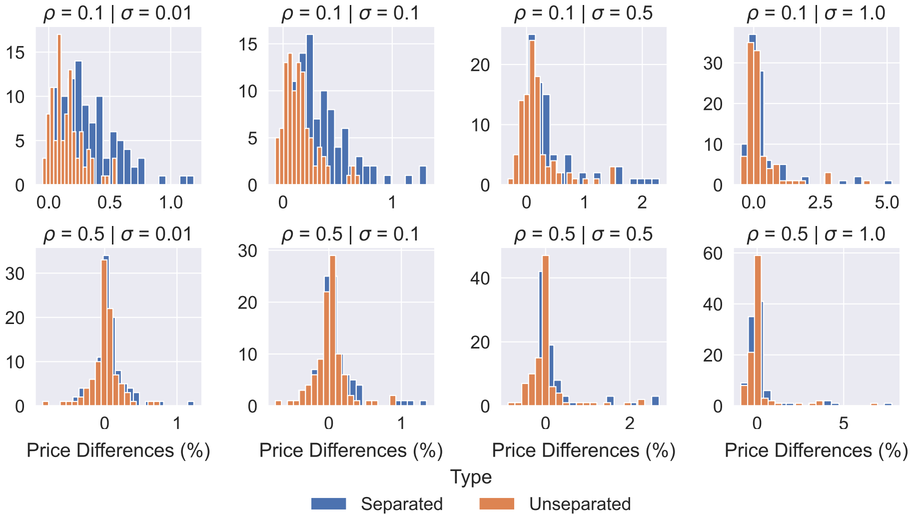
Table: Static Model Parameterization

Parameter	Symbol	Value
Number of firms	N	99
Nesting identifier	g	$g \in \mathcal{G} = \{1, 2, 3\}$
Separation of nests	s	$s \in S = \{0, 1\}$
Variance of covariates	σ	$\sigma \in \Sigma = \{0.01, 0.1, 0.5, 1.0\}$
Covariate (1)	X_1	$s \times \frac{g}{2} + N(0, \sigma)$
Covariate (2)	X_2	$s \times \frac{g}{2} + N(0, \sigma)$
Price sensitivity	α	-0.5
Taste for X_1	β_1	1
Taste for X_2	β_2	0.25
Nesting parameter	ρ	$\rho \in P = \{0.1, 0.5, 0.7, 0.99\}$

Independent Nest Pricing

Simulated Data

Figure: Independent Nests



Independent Nest Pricing

Simulated Data

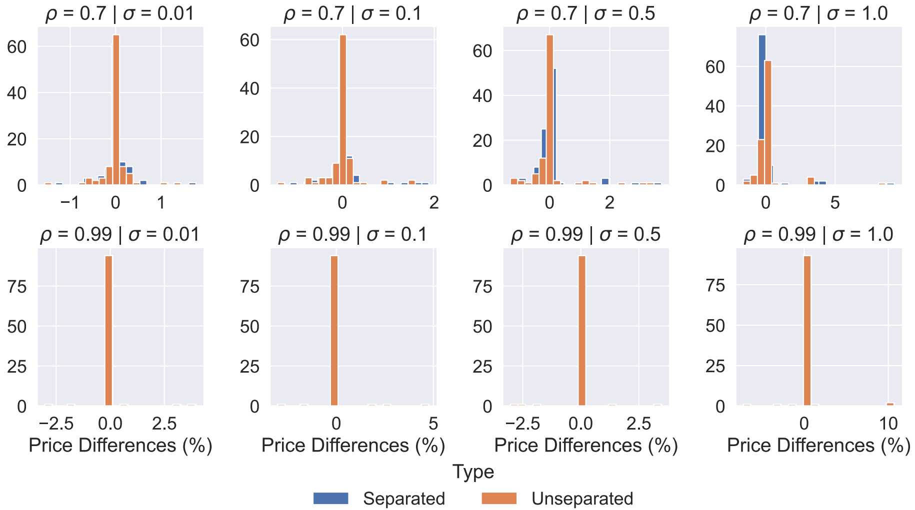
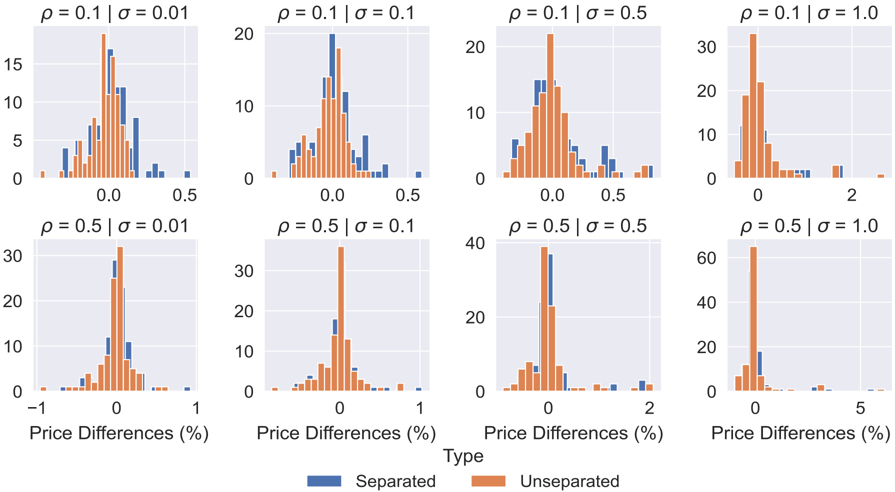


Figure: Independent Nests

Averaged Nest Pricing

Simulated Data

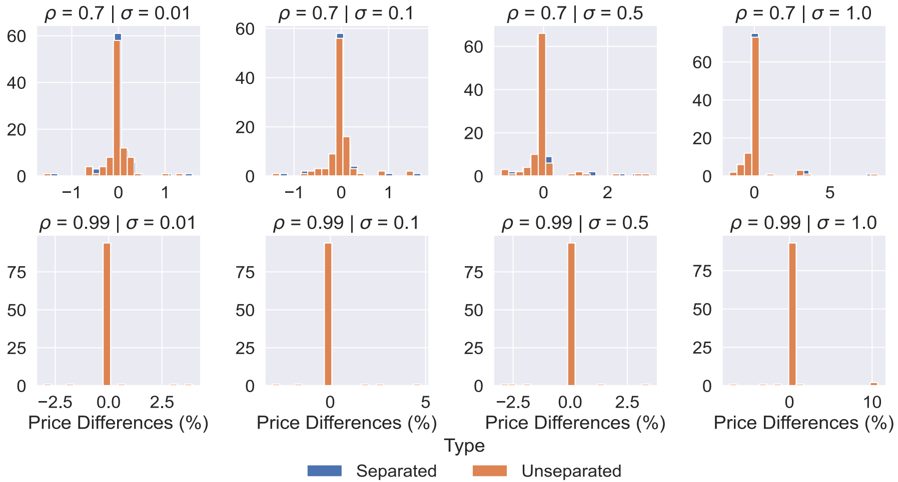
Figure: Averaged Nests



Averaged Nest Pricing

Simulated Data

Figure: Averaged Nests



Takeaways

- Approximations of the state space work well in a static setting
- They work especially well when at least one of a few conditions is met:
  - Nests are tightly defined ($\sigma \to 0$)
  - Nests are separated from one another ($s \neq 0$)
  - Within-nest substitution is strong ($\rho \to 1$)
- Averaged nest pricing dominates independent nest pricing
  - Differences are negligible when within-nest substitution is strong enough
  - Differences are also small when nests are not well separated

Dynamic Simulations [in progress]

- Follow the example of Ifrach and Weintraub (2017) with advertising[4]

$$u_{ijt} = \theta_1 \log(x_{jt} + 1) + \theta_2 \log(Y - p_{jt}) + G(\mathbf{X}_j) + \nu_{ig(j)t} + (1 - \rho)\nu_{ijt}$$

- Consumers have preferences over quality x and price p
- They also collapse high-dimensional characteristics into a nest fixed effect $G(\mathbf{X})$
- Transition probabilities are as denoted earlier

[4] Assume that advertising is not informative...

Dynamic Simulations [in progress]

Parameter	Symbol	Value
Discount factor	β	0.95
Market size	m	50
Demand quality weight	θ_1	0.5
Demand composite good weight	θ_2	0.5
Demand nest fixed effects	$G(\mathbf{X})$	$\{0.01, 0.02\}$
Demand nesting parameter	ρ	0.9
Average income	Y	1
Investment efficacy	a	3
Depreciation probability	δ	0.7
Entry state	x^e	10
Marginal investment cost	d	0.5
Marginal cost	c	0.5
Sunk entry cost	κ	35

Roadmap

Conclusions

- The curse of dimensionality remains a large problem in dynamic problems
- Methods such as oblivious, partially oblivious, and moment-based equilibria help
- However, it is difficult to incorporate high-dimensional heterogeneity

- Progress thus far:
  - Proposed approaches nest original solution concepts
  - Static simulations show when the assumptions required are reasonable
- *Next*: Comparison of Markov perfect equilibrium, original methods, and new methods

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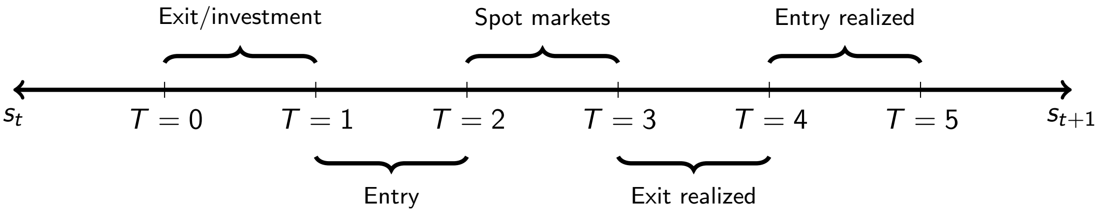
Assumptions Back

1. For all x, $\Pi(x, s)$ is increasing in x and $\sup_{x,s} \Pi(x, s) < \infty$
2. Scrap values ϕ_{jt} are i.i.d. and $\mathbb{E}[\phi_{jt}] < \infty$
3. Transition probability $P(x_{j,t+1}|x_{jt} = x, \iota)$ is i.i.d. across firms and independent of ϕ_{jt}
4. Entrants' mass M_t is a Poisson random variable and is conditionally independent of (ϕ_{jt}, x_{jt}) (conditional on s_t)
5. $\kappa > \beta\bar{\phi}$ where $\bar{\phi}$ is the expected net present value of entering the market, earning zero profits each period, and then exiting at an optimal stopping rule

Model Timing

Back

Figure: Model Timing



Partially Oblivious Value Function

Back

- Denote $\bar{D}$ as the set of indices associated with dominant firms
- Extend the state space to $\bar{x}_{it} = \left(x_{it}, \mathbb{1}\{i \in \bar{D}\}\right)$
- Partially oblivious strategies and value function:

$$\tilde{V}_p(\bar{x}, w|\mu', \mu, \lambda) = \mathbb{E}_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}(\Pi_{ik} - c(\iota_{ik}, x_{ik})) + \beta^{\tau_i - t}\phi_{i,\tau_i} \middle| \bar{x}_{it} = \bar{x}, w_t = w\right]$$

Perceived Value Function

Back

- Firms use a function θ that maps fringe firms' states into moments
- Firms use a perceived transition kernel to describe the evolution of the industry
- A joint transition kernel also incorporates transitions from fringe to dominance
- Firms maximize a perceived value function over the perceived joint transition kernel:

$$\tilde{V}_m(x, \hat{s}|\mu', \mu, \lambda) = \mathbb{E}_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}(\Pi(x_{ik}, \hat{s}_k) - c(\iota_{ik}, x_{ik})) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle|\bar{x}_{it} = \bar{x}, \hat{s}_t = \hat{s}\right]$$

Nested Oblivious Equilibrium Back

Proposition (1).

Consider the case when $K = 1$. Cases (i) and (ii) each constitute an oblivious equilibrium. Now consider the case when $K = N$. In Case (iv), firms are not oblivious to themselves but are oblivious to all other firms. Thus, each of these cases comprises an oblivious equilibrium. □

Proposition (2).

Consider the case when $K = 1$. Cases (iii) and (iv) converge to Markov perfect equilibrium because all firms are in the same nest and are not oblivious to each other. □

Nested Oblivious Equilibrium

Back

Algorithm Nested Oblivious Equilibrium

```
Compute a set of nests 𝒢 via clustering or a priori specification
Augment the state space x according to Cases (i)-(iv)
Initialize n = 1, Δ = ε + 1, μ^g, λ^g for all g ∈ 𝒢
for g ∈ 𝒢 do
    Initialize n = 1, Δ = ε + 1
    while Δ > ε do
        Solve Ṽ(·) for each state x and nest g
        Update λ^g := λ^g + ((λ^g)' − λ^g)/(1 + n^σ) for all g ∈ 𝒢
        Update μ^g := μ^g + ((μ^g)' − μ^g)/(1 + n^σ) for all g ∈ 𝒢
        n = n + 1
    end while
end for
return {(λ^g, μ^g)} for all g ∈ 𝒢
```

Nested Partially Oblivious Equilibrium Back

Proposition (3).

Suppose there are no dominant firms. This is exactly nested oblivious equilibrium. Now suppose that all firms are dominant. Case (i) is thus equivalent to Case (iii) of nested oblivious equilibrium and Case (ii) is the same as Case (iv) of nested oblivious equilibrium because firms respond to the states of all firms within their nest and are oblivious to firms outside their nest. Propositions 1 and 2 show that oblivious equilibrium and Markov perfect equilibrium are subsequently nested.

Suppose that there is at least one dominant firm and $K = 1$. This is exactly the solution concept of partially oblivious equilibrium. □

Nested Partially Oblivious Equilibrium Back

Algorithm Nested Partially Oblivious Equilibrium

Compute a set of nests $\mathcal{G}$ via clustering or *a priori* specification
Augment the state space x according to Cases (i) and (ii)
Initialize $n = 1$, $\Delta = \varepsilon + 1$, $\mu^g(\bar{x}, w)$, $\lambda^g(w)$ for all $g \in \mathcal{G}$, $\bar{x}$, and w
for $g \in \mathcal{G}$ **do**
    Initialize $n = 1$, $\Delta_1 = \varepsilon_1 + 1$, $\Delta_2 = \varepsilon_2 + 1$
    while $\Delta_1 > \varepsilon_1$, $\Delta_2 > \varepsilon_2$ **do**
        Compute $\tilde{f}^g(w|\mu, \lambda)$ for all w
        Solve $\tilde{V}_p(\cdot)$ for each state $\bar{x}$, w in nest g with $(\mu^g)'$
        $\lambda^g(w)' = \lambda^g(w)\left(\beta\mathbb{E}\left[\tilde{V}_p((x^e, 0), w_{t+1}|(\mu^g)', \mu^g, \lambda^g)|w_t = w\right]/\kappa\right)\ \forall w$
        Update errors and policy functions
    end while
end for
return $\{(\lambda^g, \mu^g)\}$ for all $g \in \mathcal{G}$

Nested Moment-Based Markov Equilibrium

Back

Proposition (4).

Suppose that the moment function is the identity $\theta(f) = f$. Firms respond obliviously to the relevant histograms of fringe firms and dominant firms. This corresponds to (simulated) nested partially oblivious equilibrium. Proposition 2 shows that all solution concepts are nested.

Now suppose that $K = 1$. All firms are contained in the same nest, meaning they each respond to their own state and the states of the dominant firms, and use moment-based strategies to respond to the fringe. This constitutes a moment-based Markov equilibrium. □

Nested Moment-Based Markov Equilibrium

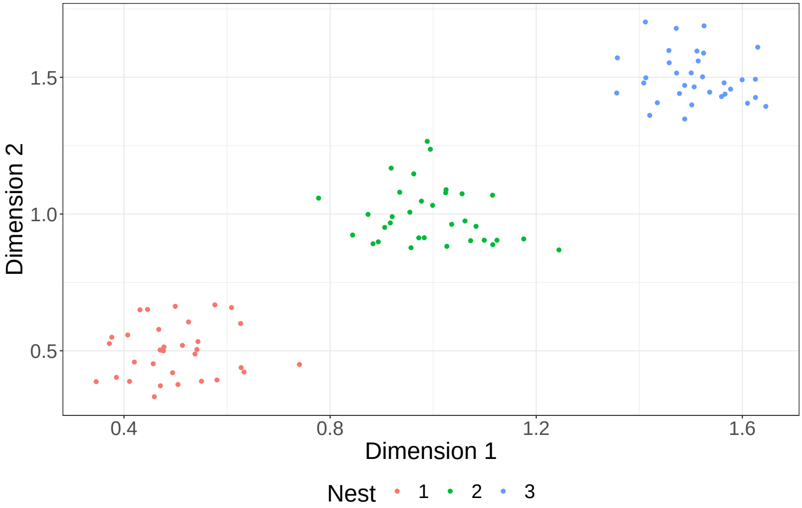
Back

Algorithm Nested Moment-Based Markov Equilibrium

1: Compute a set of nests $\mathcal{G}$ via clustering or *a priori* specification
2: Set initial industry state $s_0 = \{(f_0^g, d_0^g)\}_{g \in \mathcal{G}}$, initialize $n = 1$, $\Delta = \varepsilon + 1$
3: **for** $g \in \mathcal{G}$ **do**
4: Initialize $n = 1$, $\Delta = \varepsilon + 1$
5: **while** $\Delta > \varepsilon$ **do**
6: Simulate $s_t^g = (f_t^g, d_t^g, z_t)$ with $\hat{s}_t^g$ for $t \in \{1, ..., T\}$ and $g \in \mathcal{G}$
7: Compute frequency of states and perceived transition kernel
8: Solve $\tilde{V}_m(\cdot)$ for each state $x, \hat{s}, g \in \mathcal{G}$ with $(\mu^g)'$
9: $\lambda^g(\hat{s}^g)' = \beta \mathbb{E}_{\mu^g, \lambda^g} \left[\tilde{V}_m(x^e, \hat{s}_{t+1}^g | (\mu^g)', \mu^g, \lambda^g) | \hat{s}_t^g = \hat{s}^g \right] \ \forall \hat{s}^g, \ g \in \mathcal{G}$
10: Update errors and policy functions
11: **end while**
12: **end for**
13: **return** $\{(\lambda^g, \mu^g)\}$ for all $g \in \mathcal{G}$

Static Data-Generating Process Back

Figure: Data-Generating Process with $s = 1$, $\sigma = 0.1$



Static Data-Generating Process

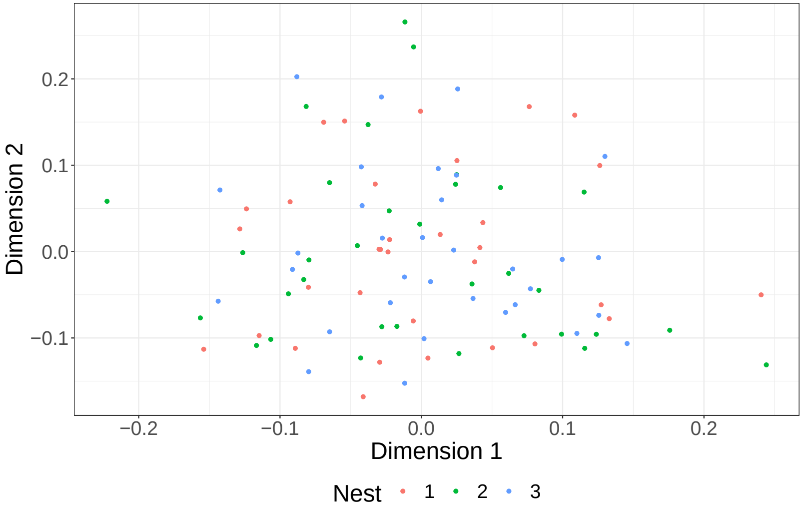


Figure: Data-Generating Process with $s = 0$, $\sigma = 0.1$